This completes our evaluation of $J(a)$ for $0 \le a \le \sqrt{3}$. Evaluation of $J(a)$ for a generic a involves six evaluations of the Lobachevsky function.

Editorial comment. Maple evaluates the Lobachevsky function in terms of a dilogarithm of a complex argument. Some solvers evaluated $J(a)$ directly in terms of complex dilogarithms. Both John Melville and the GCHQ Problem Solving Group noted that $J(1) = (\pi/2)\log((1+\sqrt{2})/2) - G$, where G is Catalan's constant.

Both parts also solved by W. Chu (Italy), J. Grivaux (France), A. Stadler (Switzerland), GCHQ Problem Solving Group (U. K.). Part (a) only also solved by A. Ahmed (India), R. Bagby, M. Bataille (France), M. Benito & Ó. Ciaurri & E. Fernández (Spain), P. Bracken, R. Chapman (U. K.), J. Clark, P. P. Dályay (Hungary), J. A. Grzesik, G. L. Isaacs, S. E. Louridas (Greece), J. Melville (U. K.), D. Shelupsky, R. Stong, D. B. Tyler, R. Zarnowski, L. Zhou, NSA Problems Group, and the proposer.

Integral over a Simplex

11039 [2003, 743]. *Proposed by Noah Rosenberg, University of Southern California, Los Angeles, CA, and Richard Stong, Rice University, Houston, TX.* Let $\Delta_k = \{(x_1,\dots,x_k)\colon x_i \ge 0 \text{ and } \sum_{i=1}^n x_i \le 1\}$ and define x_{k+1} on Δ_k by $x_{k+1} = 1 - \sum_1^k x_i$. Suppose that $a_1,\dots,a_{k+1}$ are distinct real numbers and that f is a k-times differentiable function on the interval $[\min(a_i), \max(a_i)]$. Prove that

$$\int_{\Delta_k} f^{(k)}\left(\sum_1^{k+1} a_i x_i\right) dx_1 \dots dx_k = D_f/D,$$

where

$$D = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \dots & a_{k+1}^{k-1} \\ a_1^k & a_2^k & \dots & a_{k+1}^k \end{vmatrix},$$

and where D_f is the same as D but with the last row replaced by $(f(a_1),\dots,f(a_{k+1}))$.

Composite solution by Richard Bagby, New Mexico State University, Las Cruces, NM, Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy, and the editors. By Cramer's rule, D_f/D is the leading coefficient of the unique polynomial P of degree k that satisfies $P(a_i) = f(a_i)$ for $1 \le i \le k+1$. By Newton's form of the interpolating polynomial, the leading coefficient is the divided difference $f(a_1,\dots,a_{k+1})$, where $f(a_1,a_2),\dots,f(a_1,\dots,a_{k+1})$ are defined by $f(a_1,a_2) = \frac{f(a_1)-f(a_2)}{a_1-a_2}$ and for $m \ge 2$ by

$$f(a_1,\dots,a_{m+1}) = \frac{f(a_1\dots,a_m) - f(a_2,\dots,a_{m+1})}{a_1 - a_{m+1}}.$$

Let $I(a_1,\dots,a_{k+1})$ denote the value of the integral. Then

$$I(a_1,a_2) = \int_0^1 f'(a_1x_1 + a_2(1-x_1))\,dx_1 = \frac{f(a_1)-f(a_2)}{a_1-a_2}.$$

For $m \ge 2$, let $y_i = x_i$ for $i = 2,\dots,m$ and set $y_1 = y_{m+1} = 1 - \sum_{i=2}^m y_i$. Thus $x_1 + x_{m+1} = y_1$. First integrating with respect to x_1, we obtain the following series of expressions for $I = I(a_1, \dots, a_{m+1})$:

$$
\begin{aligned}
I &= \int_{\Delta_m} f^{(m)}\left(\sum_{i=1}^{m+1} a_i x_i\right) dx_1 \dots dx_m \\
&= \int_{\Delta_{m-1}} dy_2 \dots dy_m \int_0^{y_1} f^{(m)}\left(\sum_{i=2}^{m} a_i y_i + a_1 x_1 + a_{m+1}(y_1 - x_1)\right) dx_1 \\
&= \frac{1}{a_1 - a_{m+1}} \int_{\Delta_{m-1}} f^{(m-1)}\left(\sum_{i=1}^{m} a_i y_i\right) dy_2 \dots dy_m \\
&\qquad - \frac{1}{a_1 - a_{m+1}} \int_{\Delta_{m-1}} f^{(m-1)}\left(\sum_{i=2}^{m+1} a_i y_i\right) dy_2 \cdots dy_m \\
&= \frac{I(a_1, \dots, a_m) - I(a_2, \dots, a_{m+1})}{a_1 - a_{m+1}}.
\end{aligned}
$$

Hence $I(a_1, \dots, a_{k+1}) = f(a_1, \dots, a_{k+1}) = D_f/D$.

Also solved by S. Amghibech (Canada), D. Beckwith, R. Chapman (U.K.), W. Chu & M. Pierluigi (Italy), J.-P. Grivaux (France), J. A. Grzesik, E. A. Herman, J. C. Hickman, J. H. Lindsey II, O. P. Lossers (Netherlands), D. Shelupsky, L. Zhou, V. Zoltán (Hungary), BSI Problems Group (Germany), Szeged Problems Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), and the proposers.

Range Identities for Hermitian Idempotent Matrices

11040 [2003, 843]. *Proposed by Yongge Tian, Queen's University, Kingston, Ontario, Canada.* Let A and B be $n \times n$ complex matrices such that $A = A^2 = A^*$ and $B = B^2 = B^*$. That is, A and B are both idempotent and Hermitian. Show that

$$\operatorname{range}[(AB)^2 - (BA)^2] = \operatorname{range}(ABA - BAB) = \operatorname{range}(AB - BA).$$

Solution by BSI Problems Group, Bonn, Germany. Let $C = A + B - I$ and $D = A - B$, where I is the identity matrix. We compute $AB - BA = DC$, $ABA - BAB = DC^2$, and $(AB)^2 - (BA)^2 = DC^3$. Since A and B are self-adjoint, C is selfadjoint, hence $\operatorname{range}(C) = \operatorname{range}(C^2) = \operatorname{range}(C^3)$, for there is a basis of the column space of C consisting of eigenvectors of C. The claim follows.

Editorial comment. The range identities hold for bounded linear operators on a complex Hilbert space.

Also solved by F. Boca, R. Chapman (U. K.), A. Coffman, K. Dale (Norway), A. Dove, L. M. DeAlba, M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), J. Kim, J. Kim & A. Zotov, O. Krafft & M. Schaefer (Germany), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

An Invariance of Variance over Expectation

11042 [2003, 843]. *Proposed by M. N. Deshpande, Institute of Science, Nagpur, India, and Kavita Laghate, S. N. D. T. Women's University, Mumbai, India.* Let n and k be positive integers with $1 \le k < n$, let $[n]$ denote $\{1, 2, \dots, n\}$, and let $\mathcal{P}$ be the discrete probability space consisting of the k-element subsets of $[n]$, each equally likely. For $S \in \mathcal{P}$, let π_S be the permutation of $[n]$ obtained by listing S in increasing order followed by $[n] - S$ in increasing order. Let $X(S)$ be the number of inversions in π_S (that